A Lipschitz stable reconstruction formula for the wave speed from boundary measurements

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(Joint work with Lauri Oksanen)

Problem Formulation

- Consider the wave equation

$$\begin{cases} \partial_t^2 u(x,t) - c(x)^2 \Delta u(x,t) = 0 & \text{in } M \times (0,\infty) \\ u(x,0) = u_t(x,0) = 0 & \text{in } M \\ u(x,t) = f(x,t) & \text{in } \partial M \times (0,\infty) \end{cases}$$

where $M \subset \mathbb{R}^n$ is a bounded domain with smooth boundary ∂M, $c(x)$ is a strictly positive smooth wave speed and f is Dirichlet boundary condition (control).

- We denote the solution of the wave equation by $u^f(x,t) = u(x,t)$ and define the Dirichlet-to-Neumann operator, which models the boundary measurements,

$$\Lambda_{c,T} : f \mapsto \frac{\partial u^f}{\partial \nu}\Big|_{\partial M \times (0,T)}, \quad T > 0.$$

- $\Lambda_{c,T}$ is continuous $H^1_{cc}(\partial M \times (0,T)) \to L^2(\partial M \times (0,T))$, where $H^1_{cc}(\partial M \times (0,T)) = \{f \in H^1(\partial M \times (0,T));\ f(x,0) = 0\}$

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Problem Formulation

- Inverse problem: Reconstruct the wave speed $c(x)$ on M from the knowledge of the Dirichlet-to-Neumann operator $\Lambda_{c,T}$.
- Sufficient large T: By the finite speed of propagation for the wave equation, if there is $x_0 \in M$ such that $T < 2d(x_0, \partial M)$, where d is the distance function of the Riemannian manifold $(M, c^{-2}dx^2)$, then $\Lambda_{c,T}$ can not contain any information about $c(x_0)$.
- Uniqueness: The inverse problem is uniquely solvable, i.e.

$$\Lambda_{c,T} = \Lambda_{\tilde{c},T} \Longrightarrow c(x) = \tilde{c}(x),$$

for T satisfying $T > \max_{x \in M} 2d(x, \partial M)$. This can be proved by either using the boundary control (BC) method or by using the complex geometric optics (CGO) solutions. However, these methods normally only give logarithmic type stability.

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Literature on Stability

- Hölder-type stability results were established in [Stefanov-Uhlmann 98, 05], [Bellassoued-Dos Santos 11] based on the simplicity assumption on the geometry. Especially the latter gives an explicit Hölder exponent $1/2$, however, the technique does not give a global reconstruction method.
- Hölder-type stability with an exponent strictly better than $1/2$ allows an inverse problem to be solved locally by the nonlinear Landweber iteration [de Hoop-Qiu-Scherzer 12]. Moreover, the convergence rate of the iteration is linear if and only if the problem is Lipschitz stable.
- For recover the potential of the wave equation from the DN map, Hölder stability [Sun 90], "almost Lipschitz" stability [Bao-Yun 09].
- If $|\Delta u(x,0)| \neq 0$, then Lipschitz-type stability can be obtained by using a single measurement: method by Carleman estimates that was originated in [Bukhgeim-Klibanov 81].

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Main Result

Theorem (Reconstruction Formula)

Suppose the wave equation is exact controllable in time $T > 0$. Define the harmonic exponential functions

$$\phi_{\xi,\eta}(x) = e^{(-\eta+i\xi)\cdot x/2}, \quad \psi_{\xi,\eta}(x) = e^{(\eta+i\xi)\cdot x/2},$$

where $\xi, \eta \in \mathbb{R}^n$, $|\xi| = |\eta|$ and $\xi \cdot \eta = 0$. Then

$$\mathcal{F}(c^{-2})(\xi) = (K(\Lambda_{c,2T})^{\dagger} B(\Lambda_{c,T})\phi_{\xi,\eta}, B(\Lambda_{c,T})\psi_{\xi,\eta})_{L^2(\partial M\times(0,T))},$$

where $K(\Lambda_{c,2T})$ and $B(\Lambda_{c,T})$ are operators that can be represented in terms of the Dirichlet-to-Neumann operator and $K^{\dagger}$ denotes the pseudoinverse operator of K.

Main Result

Theorem (Lipschitz Stability)

Suppose that the wave equation is stably controllable in $T > 0$, or the Riemannian manifold $(M, c^{-2}dx^2)$ admits a strictly convex function that has no critical points. Let $M \subset B(0, R)$ for some $R > 0$ and $c(x) \geq \epsilon_U > 0$, for all $x \in M$ and $c \in U$. Then there is $C > 0$ depending on M, T, c, ϵ_U such that for all $\tilde{c} \in U$

$$|\mathcal{F}(\tilde{c}^{-2} - c^{-2})(\xi)| \leq Ce^{2R|\xi|}\|\Lambda_{\tilde{c},2T} - \Lambda_{c,2T}\|_*, \quad \xi \in \mathbb{R}^n,$$

where for $\Sigma = \partial M \times (0, T)$

$$\|\Lambda_{c,2T}\|_* := \|K(\Lambda_{c,2T})\|_{L^2(\Sigma)\to L^2(\Sigma)} + \|\Lambda_{c,T}\|_{H^1_{cc}(\Sigma)\to L^2(\Sigma)}.$$

- Remark: $\|\cdot\|_*$ is indeed a norm.

Exact Controllability and Continuous Observability

- We recall the wave equation is called exactly boundary controllable from $\Gamma \subset \partial M$ in time $T > 0$ if the following control-to-solution map is surjective:

$$f \mapsto (u^f(T), u_t^f(T)) : \; L^2(\partial M \times (0,T)) \to L^2(M) \times H^{-1}(M)$$

- It is well-known that, by duality, the exact boundary controllability is equivalent to the continuous observability inequality of the dual problem. That is, there exists a constant $C_{obs} > 0$, such that

$$\|(w_0, w_1)\|_{H_0^1(M)\times L^2(M)} \leq C_{obs} \|\frac{\partial w}{\partial \nu}\|_{L^2(\Gamma\times(0,T))}$$

where w is the solution of the dual problem

$$\begin{cases} \partial_t^2 w(x,t) - c^2(x)\Delta w(x,t) = 0 & \text{in } M \times (0,T) \\ w(x,T) = w_0(x),\; w_t(x,T) = w_1(x) & \text{in } M \\ w = 0 & \text{in } \partial M \times (0,T) \end{cases}$$

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Stable Observability

Definition

We say that the wave equation is stably controllable from $\Gamma \subset \partial M$ in time $T > 0$, for $c \in U$, if there is a unified $C_{obs} > 0$ such that for all $c \in U$ the solutions of the wave equations satisfy the continuous observability inequality $\|(w_0, w_1)\|_{H_0^1(M)\times L^2(M)} \leq C_{obs} \|\frac{\partial w}{\partial \nu}\|_{L^2(\Gamma\times(0,T))}$.

Theorem

Assume that there is a strictly convex function $\ell \in C^3(M)$ with respect to the metric tensor $c^{-2}dx^2$, and that ℓ has no critical points. Let U be bounded in $C^2(M)$ and let $\Gamma \subset \partial M$ contain $\{x \in \partial M;\ \nabla\ell(x)\cdot\nu \geq 0\}$. Then there is a neighborhood V of c in $C^1(M)$ and $T > 0$ such that the wave equations are stably controllable for the wave speeds in the set $U \cap V$, from Γ in time T.

Control to Solution Map W

- We now consider the operator which maps the control f to the solution u at time T:

$$Wf := u^f(T), \quad W : L^2(\partial M \times (0,T)) \to L^2(M).$$

- Then $W^*\phi = \frac{\partial w}{\partial \nu}|_{\partial M \times (0,T)}$ with w being the solution of the dual problem with $w_0 = 0$ and $w_1 = \phi$.
- Both W and W^* are bounded linear operators with the norm $\|W\| = \|W^*\| \leq C(c)$.
- If the wave equation is exactly controllable, i.e., W is surjective, then we can consider the pseudoinverse of W: $W^\dagger\phi$ gives the minimum norm control that solves the control equation $Wf = \phi$.
- By the observability inequality we can get $\|W^\dagger\| = \|(W^\dagger)^*\| \leq C_{obs}$ and $\|(W^*W)^\dagger\| \leq C_{obs}^2$.

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Blagoveščenskiĭ Type Identities

Lemma

Let $f, h \in C_0^\infty(\partial M \times (0,T))$. *Then*

$$(u^f(T), u^h(T))_{L^2(M;c^{-2}(x)dx)} = (f, K(\Lambda_{c,2T})h)_{L^2(\partial M \times (0,T))}.$$

where the operator $K(\Lambda_{c,2T})$ *is defined by*

$$K(\Lambda_{c,2T}) := R\Lambda_{c,T}RJ\Theta - J\Lambda_{c,2T}\Theta,$$

where R *is the time reversal on* $(0,T)$, Θ *is the extension by zero from* $(0,T)$ *to* $(0,2T)$ *and*

$$Jf(t) := \frac{1}{2}\int_t^{2T-t} f(s)ds, \quad f \in L^2(0,2T), \ t \in (0,T).$$

- [Bingham-Kurylev-Lassas-Siltanen 08]

Blagoveščenskiĭ Type Identities

Lemma

Let $f \in C_0^\infty(\partial M \times (0,T))$ *and let* ϕ *be a harmonic function. Then*

$$(u^f(T), \phi)_{L^2(M;c^{-2}(x)dx)} = (f, B(\Lambda_{c,T})\phi)_{L^2(\partial M \times (0,T))}.$$

where the operator $B(\Lambda_{c,T})$ *is defined by*

$$B(\Lambda_{c,T}) := R\Lambda_{c,T}RI\mathcal{T}_0 - I\mathcal{T}_1,$$

$\mathcal{T}_j$, $j = 0, 1$, *are the first two traces on* ∂M, *that is* $\mathcal{T}_0\phi = \phi|_{\partial M}$ *and* $\mathcal{T}_1\phi = \frac{\partial \phi}{\partial \nu}|_{\partial M}$, *and*

$$If(t) := \int_t^T f(s)ds, \quad f \in L^2(0,T), \ t \in (0,T).$$

Inner products of harmonic functions

- The first identity implies that for $f, h \in C_0^\infty(\partial M \times (0, T))$

$$\begin{aligned}(f, W^* W h)_{L^2(\partial M \times (0,T))} &= (u^f(T), u^h(T))_{L^2(M; c^{-2}dx)} \\ &= (f, Kh)_{L^2(\partial M \times (0,T))}.\end{aligned}$$

  Thus $K = W^* W$ extends as a continuous operator on $L^2(\partial M \times (0, T))$.

- On the other hand, the second identity implies that for $f \in C_0^\infty(\partial M \times (0, T))$ and harmonic $\phi \in C^\infty(M)$

$$\begin{aligned}(f, W^* \phi)_{L^2(\partial M \times (0,T))} &= (u^f(T), \phi)_{L^2(M; c^{-2}dx)} \\ &= (f, B\phi)_{L^2(\partial M \times (0,T))}.\end{aligned}$$

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Inner products of harmonic functions

- Notice that $WW^\dagger$ is the identity on $L^2(M)$ since W is surjective. We thus have for any harmonic functions ϕ and ψ

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&= ((W^*W)^\dagger W^*\phi, W^*\psi)_{L^2(M;c^{-2}dx)} \\
&= (K^\dagger B\phi, B\psi)_{L^2(\partial M\times(0,T))},
\end{aligned}
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- By taking $\phi = \phi_{\xi,\eta}(x) = e^{(-\eta+i\xi)\cdot x/2}$ and $\psi = \psi_{\xi,\eta}(x) = e^{(\eta+i\xi)\cdot x/2}$, then we have

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Stability Estimate

Therefore for harmonic functions ϕ, ψ and another speed $\tilde{c} \in U$

$$|(\phi, \psi)_{L^2(M;\tilde{c}^{-2}dx)} - (\phi, \psi)_{L^2(M;c^{-2}dx)}|$$

$$= |(K^\dagger B\phi, B\psi) - (\tilde{K}^\dagger \tilde{B}\phi, \tilde{B}\psi)|$$

$$\leq |(K^\dagger B\phi, B\psi) - (\tilde{K}^\dagger B\phi, B\psi)| + |(\tilde{K}^\dagger B\phi, B\psi) - (\tilde{K}^\dagger B\phi, \tilde{B}\psi)|$$

$$+ |(\tilde{K}^\dagger B\phi, \tilde{B}\psi) - (\tilde{K}^\dagger \tilde{B}\phi, \tilde{B}\psi)|$$

where we have omitted writing $L^2(\partial M \times (0, T))$ as a subscript.

Stability Estimate

- Estimate each difference, recall the definition of K and B

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\begin{aligned}
&|(K^\dagger B\phi, B\psi) - (\tilde{K}^\dagger B\phi, B\psi)| \\
&= |((K^\dagger - \tilde{K}^\dagger)B\phi, B\psi)| \\
&\leq 3C_{obs}^4 \|W^*\|^2_{L^2(M)\to L^2(\Sigma)} \|\tilde{K} - K\|_{L^2(\Sigma)} \|\phi\|_{L^2(M)} \|\psi\|_{L^2(M)}
\end{aligned}
$$

- [Izumino 83] If $A, B \in \mathcal{L}(H, K)$ with closed ranges, then

$$\|B^\dagger - A^\dagger\| \leq 3\max\{\|B^\dagger\|^2, \|A^\dagger\|^2\}\|B - A\|.$$

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Stability Estimate

- $$|(\tilde{K}^\dagger B\phi, B\psi) - (\tilde{K}^\dagger B\phi, \tilde{B}\psi)| \leq |(\tilde{K}^\dagger B\phi, (B - \tilde{B})\psi)|$$
$$\leq C_{obs}^2 \|W^*\|_{L^2(M)\to L^2(\Sigma)} C \|\tilde{\Lambda}_T - \Lambda_T\|_{H^1_{cc}(\Sigma)\to L^2(\Sigma)} \|\phi\|_{L^2(M)} \|\psi\|_{C^1(\partial M)}$$

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- We hence have that there is a constant $C = C(C_{obs}, c, \epsilon_U, M, T) > 0$ such that for $\tilde{c} \in U$ and harmonic ϕ, ψ

$$\left|(\phi,\psi)_{L^2(M;\tilde{c}^{-2}dx)} - (\phi,\psi)_{L^2(M;c^{-2}dx)}\right|$$

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$$= \|\tilde{\Lambda}_{2T} - \Lambda_{2T}\|_* \|\phi\|_{C^1(M)}\|\psi\|_{C^1(M)}$$

- Let $R > 0$ such that $M \subset B(0,R)$, again by taking the harmonic functions

$$\phi(x) := e^{(-\eta+i\xi)\cdot x/2}, \quad \psi(x) := e^{(\eta+i\xi)\cdot x/2}.$$

Then we get

$$\begin{aligned} \left|\mathcal{F}\left(\tilde{c}^{-2} - c^{-2}\right)(\xi)\right| &= \left|(\phi,\psi)_{L^2(M;\tilde{c}^{-2}dx)} - (\phi,\psi)_{L^2(M;c^{-2}dx)}\right| \\ &\leq Ce^{2R|\xi|}\|\tilde{\Lambda}_{2T} - \Lambda_{2T}\|_* \quad \tilde{c} \in U,\ \xi \in \mathbb{R}^n. \end{aligned}$$

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Summary

- Reconstruct wave speed from Dirichlet-to-Neumann map: combine the BC method and the CGO solutions method.
- Exact controllability $\Rightarrow$ Reconstruction formula.
- Stable controllability $\Rightarrow$ Local Lipschitz-type stability.
- Stability estimate works for low frequency.

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Thank you!